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Simple Eddy Viscosity Relations for Three-Dimensional Turbulent Boundary Layers

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THE technique whereby the Clauser[1] relation for an integral eddy viscosity,

$$\nu_{to} = K\int_0^\infty (u_e - u)\mathrm{d}y \tag{1}$$

forms the basis of an outer function which is comported with an inner function to describe turbulent boundary layers has now been extensively applied in many engineering research and design studies since it was first introduced[2] (in Ref. 3, a composite function of inner and outer functions is formally described). The technique has limitations but at the same time is a simple basis for surprisingly good numerical predictions of turbulent boundary-layer behavior. Equation (1) was originally conceived by Clauser for two-dimensional boundary layers where $u = u(x,y)$ and $u_e \equiv u(x,\infty)$.

For three-dimensional boundary layers, Eq. (1) has been extended by Cebecci et al.[4] (where the reader will find other relevant references) to read

$$\nu_{to} = K\int_0^\infty [(u_e^2 + w_e^2)^{1/2} - (u^2 - w^2)^{1/2}]\mathrm{d}y \tag{2}$$

Here (u,w) is the horizontal velocity vector with components in the (x,z) directions. Although Eq. (2) does reduce to Eq. (1) for 2-D flow, it cannot be correct for 3-D flow since it is not invariant to a Galilean transformation as is the turbulent field it is supposed to represent. To see this in a simple case, consider a stationary, mean flowfield whose properties are invariant in the z-direction. Then, there is no reason why the flow cannot be described in another coordinate system translating with respect to the first such that $\tilde{u} = u$ and $\tilde{w} = w - W$ where W is the (transverse) velocity of the second coordinate system. Then if

$$\nu_{to} \equiv K\int_0^\infty [(u_e^2 + w_e^2)^{1/2} - (u^2 + w^2)^{1/2}]\mathrm{d}y$$

and

$$\tilde{\nu}_{to} \equiv K\int_0^\infty [(\tilde{u}_e^2 + \tilde{w}_e^2)^{1/2} - (\tilde{u}^2 + \tilde{w}^2)^{1/2}]\mathrm{d}y$$

we would have $\nu_{to} \neq \tilde{\nu}_{to}$; in other words, the solution would depend on W which is quite arbitrary.

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All of this suggests that, instead of Eq. (2), one should write

$$\nu_{to} = K\int_0^\infty y\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right]^{1/2}\mathrm{d}y \tag{3}$$

which is invariant to a Galilean transformation. Furthermore, for two-dimensional flows where $\partial u/\partial y$ is monotonic, an integration of Eq. (3) by parts will yield Eq. (1). Thus Eq. (3) is a more general interpretation of Clauser's integral, eddy viscosity relation which can, in principle, accommodate three-dimensional flows and flows where $\partial u/\partial y$ is not monotonic such as wall jets.

An interpolation formula that has come to our attention (Blackadar[5]) to provide an outer continuous function and which might be a slight improvement over the older discontinuous function is

$$\nu_t = \frac{\hat{p}}{1 + \nu_p/\nu_{to}}, \quad \nu_p \equiv \kappa^2 y^2\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] \tag{4a,b}$$

ν_t is the turbulent viscosity in the expression $(-\overline{u'v'}, -\overline{w'v'}) = \nu_t\,(\partial u/\partial y, \partial w/\partial y)$ and ν_p is the Prandtl mixing length expression. Solutions obtained from Eq. (4) may be matched to the law of the wall or an empirical viscous correction function of the form, $\nu\, f(\nu_p/\nu)$ where ν is the molecular viscosity and $f \sim 0$ as $\nu_p/\nu \to \infty$, may be added to Eq. (4a). Equivalently, another viscous function may be invented to multiply Eq. (4a) and which limits to unity as $\nu_p/\nu \to \infty$.

An Alternate Approach

In recent times, when we have wished to make a boundary-layer calculation with a simple eddy viscosity approach, we have used for the outer function

$$\nu_t = q\,\ell\,S_m \tag{5}$$

where

$$q^3 = B_l\,\ell\left[\nu_t\left\{\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right\}\right] \tag{6}$$

and for the length scale,

$$\ell = \left[\frac{(\kappa y)^4}{1 + (\kappa y/\ell_o)^4}\right]^{1/4}; \quad \ell_o \equiv \alpha\frac{\int_0^\infty |y|q\,\mathrm{d}y}{\int_0^\infty q\,\mathrm{d}y} \tag{7a,b}$$

Equation (6) is a model equation representing a balance between turbulent production and dissipation and, in situations where such a balance is expected to be true, Eq. (6) should provide a good estimate of the turbulent kinetic energy, $q^2/2$. Furthermore, Eqs. (5-7) represent a first step up the ladder to more sophisticated (and complicated) models (Mellor and Herring[6]; Mellor and Yamada[7]; Briggs, Mellor, and Yamada,[8] Hanjalic and Launder[9]).

In the above model, we have determined that $S_m = B_l^{-1/3}$, $B_l = 16.6$, and, of course, $\kappa = 0.40$. As in the case of K in Eqs. (1) or (3), α in Eq. (7a) is problem dependent; for conventional boundary layers $\alpha \simeq 0.19$. One more independent empirical constant is required compared to those in Eqs. (3) and (4) but more information is provided. Furthermore, Eqs. (5-7) may be extended rather simply to include effects of wall curvature, Coriolis acceleration, and gravity in a density stratified field (So[10], Mellor[11]). For example, for two-dimensional flow with longitudinal radius of curvature, r, it is found in Ref. 11 that

$$S_m = B_l^{-1/3}\left[1 - \frac{36A_l^2}{B_l^{2/3}}\frac{R_c}{(1-R_c)^2}\right] \tag{8}$$

where $A_l = 0.92$, $R_c \equiv 2u\{\partial(ur)/\partial y\}^{-1}$. The term in square brackets in Eq. (6) must be amended to read $[\nu_t(\partial u/\partial y -$

$u/r)^2$]. More complicated relations may be obtained for three-dimensional flow.

Solutions obtained using Eqs. (5-7) may be matched to the law of the wall or a viscous correction may be invented to modify Eq. (5). One way is to multiply the right hand side of Eq. (5) by $(g\ell/\nu)^2[(g\ell/\nu)^2+28^2]^{-1}$ and add ν to ν_t in Eq. (6). This is a completely empirical prescription, of course, but data is reproduced quite well in the viscous sublayer as well as in the outer flow region.

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Finite Element Approach to the Viscous Incompressible Flow around a Circular Cylinder

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Introduction

THE steady flow of a viscous incompressible fluid around a circular cylinder is examined for Reynolds numbers up to approximately 40. Thus it is a thoroughly laminar flow, having a bubble of recirculation behind the cylinder for Reynolds numbers higher than about 7. Cases at Reynolds numbers above 40 are usually not computed using the steady equations because unsteady vortex shedding starts to develop.

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Numerical results for the steady flow—apparently almost every new method or variant thereof is tested against this problem; two reasons are prominent for this fact: a wealth of data exists from both numerical and experimental investigations; and the problem, by no means simple, is still of such dimensions that a good accuracy can be achieved. The investigators used at first two methods: the finite difference method[1] and the method of truncated series.[2-3] A third method has come into prominence, the finite element method, and it was also applied to the solution of this problem.[4-6] Unfortunately, however, its applications to fluid dynamics were often an extension of its development in solid mechanics and available results show only crude solutions, thus shedding little or no light at all on its comparison with respect to other methods.

Within the spectrum of all the work done till now, other aspects vary too. As for example the form in which the Navier-Stokes equations were solved. Some are typical of the two-dimensional flow solutions: having as variables the streamfunction and the vorticity or the streamfunction alone. Other forms use the physical variables. Another point of importance is whether the equations are solved in the physical plane or in a transformed plane.

A last characteristic for the fast convergence of results is the representation of the far field. The usual techniques involve either a transformation bringing infinity to a finite distance or imposing the uniform flow at a finite distance or matching on the boundary the numerical solution with an asymptotic solution.

Experimental results—reliable results are available for this range of Reynolds number.[7] The drag coefficient was measured and the flow patterns were photographed. A more detailed description of the present work may be found in Ref. 8.

The Method

Equation—because of the hypotheses that characterize the flow, the Navier-Stokes equations are reduced to the continuity and two momentum equations:

$$\partial_j u_j = 0 \qquad (i,j=1,2)$$
$$\rho u_i \partial_i u_j = -\partial_j p + \mu \partial_i \partial_i u_j$$

Since μ is considered constant, cross-differentiating and substracting the momentum equations and defining the streamfunction ψ from the continuity equation will yield a biharmonic-type of equation:

$$\frac{1}{Re}\nabla^4\psi + \psi_x \nabla^2 \psi_y - \psi_y \nabla^2 \psi_x = 0 \qquad (1)$$

where Re is the Reynolds number (based on the diameter of the cylinder) and the subscripts indicate differentiation. No transformation of the physical space is performed.

Galerkin method—assuming for the streamfunction a trial function of the form $\psi' = \psi_j N_j(x,y)$ where the ψ_j are the generalized variables and the N_j the shape functions, this procedure[9] together with the use of Green's theorem yields the following set of nonlinear algebraic equations:

$$\frac{1}{Re}\int_S \nabla^2 N_i \nabla^2 N_j \psi_j \mathrm{d}S + \int_S [N_{i,x}N_{j,y} - N_{i,y}N_{j,x}]\nabla^2 N_k \psi_j \psi_k \mathrm{d}S + \frac{1}{Re}\int_C N_{i,n}\omega \mathrm{d}s - \frac{1}{Re}\int_C N_i \omega_n \mathrm{d}s + \int_C N_i \psi_s \omega \mathrm{d}s = 0 \qquad (2)$$